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A note on integral properties of periodic gravity waves in the case of a non-zero mean Eulerian velocity

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Corrected relationships are presented between integral properties of periodic gravity surface waves in the case of a non-zero mean Eulerian velocity. Also considered is the special case of a reference frame in which the mean wave momentum vanishes.

1. Introduction

Several exact relationships between integral properties of irrotational periodic gravity surface waves of finite amplitude have been derived by Longuet-Higgins (1975). These have been extended to gravity–capillary waves by Hogan (1979) and Crapper (1979), and to gravity–capillary interfacial waves by Hogan (1981).

Some of these relationships are limited to the special case of a reference frame moving with a horizontal velocity which makes the mean Eulerian velocity vanish at each point below the level of the wave troughs. This corresponds to Stokes's first definition of phase velocity.

Cokelet (1977) derived relationships between integral properties of gravity surface waves which were claimed not to be restricted to this special reference frame. However, he applied these relationships only to waves with vanishing mean Eulerian velocity, i.e. according to Stokes's first definition of wave celerity.

The Fourier approximation method of Rienecker & Fenton (1981) has been applied by the author to the computation of finite-amplitude waves in a reference frame in which the mean wave momentum vanishes (Stokes's second definition of phase velocity). Cokelet's relationships were used for the calculation of the integral properties of the waves.

However, in all cases it appeared that the computed values for the radiation stress (or mean excess of momentum flux) and the mean energy flux became negative, which is physically unrealistic since both fluxes have to be positive in the wave propagation direction.

Recalculation of the relationships between the integral properties showed that the extended relationships provided by Cokelet are incorrect for frames of reference in which the mean velocity does not vanish. Corrected integral properties for a frame of reference with a non-zero mean Eulerian velocity are presented below.

2. Definitions

Consider two-dimensional periodic gravity waves with wavelength λ, travelling over a horizontal bottom with phase velocity c. The x-axis is taken horizontal in the propagation direction of the wave and the z-axis is directed vertically upwards. The free surface is given by $z = \eta(x,t)$ and the bottom by $z = z_b$. The fluid is assumed to be incompressible and inviscid, and the velocity (u, w) to be irrotational.

The mean surface elevation is defined by

$$\bar{\eta} = \frac{1}{\lambda}\int_0^\lambda \eta \,\mathrm{d}x.$$

Similarly, the mean velocity $\bar{u}$ at any elevation below the wave trough is defined by

$$\bar{u} = \frac{1}{\lambda}\int_0^\lambda u \,\mathrm{d}x.$$

Because the motion is irrotational, the mean velocity $\bar{u}$ is independent of the elevation at which it is computed.

The integral properties to be considered are the mean wave momentum I, kinetic energy T, potential energy V, radiation stress S_{xx}, energy flux F and square of the bed velocity $\overline{u_b^2}$. They are defined by

$$I = \overline{\int_{z_b}^{\eta} \rho u \,\mathrm{d}z}, \tag{1}$$

$$T = \overline{\int_{z_b}^{\eta} \tfrac{1}{2}\rho(u^2+w^2)\,\mathrm{d}z}, \tag{2}$$

$$V = \overline{\int_{\bar{\eta}}^{\eta} \rho g z \,\mathrm{d}z}, \tag{3}$$

$$S_{xx} = \overline{\int_{z_b}^{\eta} (p+\rho u^2)\,\mathrm{d}z - \tfrac{1}{2}\rho g h^2}, \tag{4}$$

$$F = \overline{\int_{z_b}^{\eta} [p+\tfrac{1}{2}\rho(u^2+w^2)+\rho g(z-\bar{\eta})]\,u\,\mathrm{d}z}, \tag{5}$$

$$\overline{u_b^2} = \overline{[u(x,z_b,t)]^2}, \tag{6}$$

where an overbar denotes averaging over one wavelength, ρ denotes the mass density of the fluid, g the acceleration due to gravity, p the pressure and h the mean water depth $(h = \bar{\eta} - z_b)$.

These integral properties will be related to each other and to three constant quantities defined in a frame moving with phase velocity c: the mass flux Q, the total head R and the momentum flux S, as given by

$$Q = -\int_{z_b}^{\eta} \rho(u-c)\,\mathrm{d}z, \tag{7}$$

$$R = \frac{p}{\rho g} + \frac{1}{2g}[(u-c)^2+w^2] + (z-z_b), \tag{8}$$

$$S = \int_{z_b}^{\eta} [p+\rho(u-c)^2]\,\mathrm{d}z. \tag{9}$$

3. Integral properties in a frame of reference with a non-zero mean Eulerian velocity

Longuet-Higgins (1975) has derived the following relations between integral properties in a frame of reference with non-zero mean Eulerian velocity $\bar{u}$:

$$I = \rho ch - Q, \tag{10}$$

$$T = \tfrac{1}{2}(cI - \bar{u}Q), \tag{11}$$

$$S = S_{xx} - 2cI + \rho h(c^2 + \tfrac{1}{2}gh). \tag{12}$$

In the same paper relationships are found, which are only valid in a frame of reference according to Stokes's first definition of phase velocity, i.e. with $\bar{u} = 0$:

$$\tilde{S}_{xx} = 4\tilde{T} - 3\tilde{V} + \rho\, \overline{\tilde{u}_b^2}\, h, \tag{13}$$

$$\tilde{F} = \tilde{c}(3\tilde{T} - 2\tilde{V}) + \tfrac{1}{2}\overline{\tilde{u}_b^2}\,(\tilde{I} + \rho\tilde{c}h), \tag{14}$$

$$\overline{\tilde{u}_b^2} = 2g(\tilde{R} - h) - \tilde{c}^2, \tag{15}$$

where a tilde is used to denote quantities in this particular frame of reference.

The extension of the relationships between integral properties given in (13), (14) and (15) to a frame of reference with non-zero $\bar{u}$ is straightforward when using the same techniques as Longuet-Higgins (1975), or by applying a coordinate transformation to the integral properties derived in a reference frame according to Stokes's first definition of phase velocity. Both methods have been used, and they result in the same relationships, as they should.

Here we apply a coordinate transformation, and start with (13), (14) and (15), valid in a reference frame according to Stokes's first definition.

Since the quantities Q, R and S are defined in a reference frame in which the observer is moving with the phase velocity c, and in which the motion becomes steady, they have to be independent of the frame of reference, thus

$$\tilde{Q} = Q, \quad \tilde{R} = R, \quad \tilde{S} = S.$$

Also, by Stokes's first definition of phase velocity, we have the following relation between the phase velocities in both reference frames:

$$\tilde{c} = c - \bar{u}.$$

Relations between the velocities (u, w) and $(\tilde{u}, \tilde{w})$, and between the pressures p and $\tilde{p}$ can be obtained by applying a coordinate transformation to the values of these quantities in a reference frame moving with the phase velocity. The results are

$$\tilde{u}(\tilde{x}, \tilde{z}, t) = u(x, z, t) - \bar{u},$$

$$\tilde{w}(\tilde{x}, \tilde{z}, t) = w(x, z, t),$$

$$\tilde{p}(\tilde{x}, \tilde{z}, t) = p(x, z, t),$$

with the following relations between the coordinates:

$$\tilde{x} = x - \bar{u}t, \quad \tilde{z} = z.$$

From the definitions (1)–(6) the following relations between integral properties in both reference frames are easily established:

$$\tilde{I} = I - \rho\bar{u}h, \tag{16}$$

$$\tilde{T} = T - \bar{u}I + \tfrac{1}{2}\rho\bar{u}^2h, \tag{17}$$

$$\tilde{V} = V, \tag{18}$$

$$\tilde{S}_{xx} = S_{xx} - 2\bar{u}I + \rho\bar{u}^2h, \tag{19}$$

$$\tilde{F} = F - \bar{u}(S_{xx} + T + V) + \tfrac{3}{2}\bar{u}^2I - \tfrac{1}{2}\rho\bar{u}^3h, \tag{20}$$

$$\overline{\tilde{u}_b^2} = \overline{u_b^2} - \bar{u}^2. \tag{21}$$

Inserting (17), (18), (19) and (21) into (13) leads, after rearranging some terms, immediately to the following relationship:

$$S_{xx} = 4T - 3V + \rho\overline{u_b^2}h - 2\bar{u}I, \tag{22}$$

also valid in a frame of reference with $\bar{u} \neq 0$.

Equations (10), (11) and (22) are used to rearrange (20) to the following form:

$$\tilde{F} = F - \bar{u}(3T - 2V + cI + \rho\overline{u_b^2}\,h) + \bar{u}^2\,(\tfrac{5}{2}I + \rho ch - \tfrac{1}{2}\rho\bar{u}h).$$

Using this equation and (14), (16), (17), (18) and (21), we arrive at the following relationship for the mean wave energy flux F:

$$F = c(3T - 2V) + \tfrac{1}{2}\overline{u_b^2}(I + \rho ch) - 2c\bar{u}I. \tag{23}$$

By combining (15), (21) and the identities $\tilde{R} = R$ and $\tilde{c} = c - \bar{u}$, we arrive at

$$\overline{u_b^2} = 2g(R - h) - c(c - 2\bar{u}). \tag{24}$$

The desired relationships between integral properties in a frame of reference with non-zero mean Eulerian velocity are (10)–(12) and (22)–(24). Equations (22), (23) and (24) differ from Cokelet's results in that the terms $(-2\rho\bar{u}ch)$, $[-\bar{u}c(I+\rho ch)]$ and $(-2\bar{u}c/g)$ have to be added to the right-hand sides of Cokelet's equations (5.15), (5.16) and (5.17), respectively.

4. Integral properties for Stokes's two definitions of phase velocity

Two special cases will be considered here, depending on the definition of phase velocity.

In Stokes's first definition of phase velocity the mean velocity $\bar{u}$ vanishes at all elevations below the wave trough. This leads to the relationships between integral properties as given by Longuet-Higgins (1975), i.e. equations (10)–(15), applying $\bar{u} = 0$ in (11).

Stokes's second definition, however, is based on the assumption that the mean wave momentum I is equal to zero, a situation that is for instance encountered in closed wave flumes. In that case it follows from (10) that $Q = \rho ch$, and the relationships become

$$T = -\tfrac{1}{2}\rho\bar{u}ch, \tag{25}$$

$$S_{xx} = 4T - 3V + \rho\overline{u_b^2}\,h, \tag{26}$$

$$F = c(3T - 2V) + \tfrac{1}{2}\rho\overline{u_b^2}\,ch, \tag{27}$$

$$S = S_{xx} + \rho h(c^2 + \tfrac{1}{2}gh), \tag{28}$$

$$\overline{u_b^2} = 2g(R - h) - c(c - 2\bar{u}). \tag{29}$$

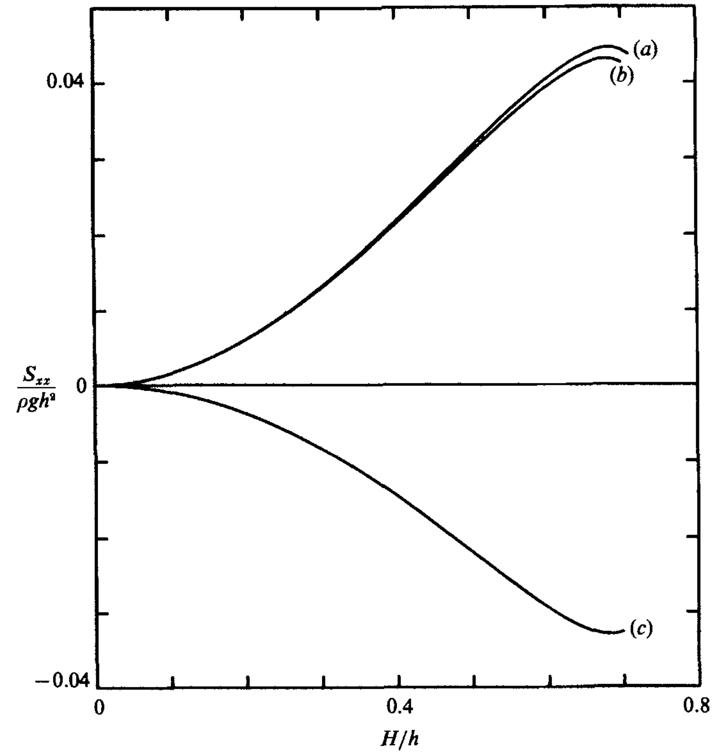

FIGURE 1. Dimensionless radiation stress $S_{xx}/(\rho g h^2)$ as a function of dimensionless wave height H/h for a dimensionless period $(\lambda/c)(g/h)^{\frac{1}{2}} = 10$. (*a*) Stokes's first definition of phase velocity, equation (5.15) of Cokelet (1977) and equation (22); (*b*) Stokes's second definition of phase velocity, equation (22); (*c*) Stokes's second definition of phase velocity, equation (5.15) of Cokelet (1977).

These equations can for instance be used to determine the integral properties in a reference frame with zero mean wave momentum I from the computed values in a reference frame with zero mean velocity $\bar{u}$. For the latter case, tabulated data are readily available in the literature (Cokelet 1977; Williams 1985).

The Fourier approximation method of Rienecker & Fenton (1981) has been used to compute the radiation stress S_{xx} and the mean energy flux F as a function of wave height H, according to both definitions of wave celerity and for a fixed dimensionless wave period $(\lambda/c)(g/h)^{\frac{1}{2}} = 10$. The wave height H has been defined as the difference in water surface elevation between wave crest and trough. The number of Fourier components was equal to 64 in all computations.

The results are presented graphically in dimensionless form in figures 1 and 2. For Stokes's first definition of wave celerity, the results are identical for the integral properties presented by Cokelet (1977) and those presented in this paper. In case of Stokes's second definition for the wave celerity the resulting radiation stress S_{xx} and mean energy flux F become negative for Cokelet's relationships, whereas they remain positive in our case.

Both radiation stress S_{xx} and mean energy flux F are smaller in the case of a zero wave momentum I (second definition) than in the case of zero mean Eulerian velocity $\bar{u}$ (first definition), which has consequences in for instance the computation of wave shoaling and wave set-up.

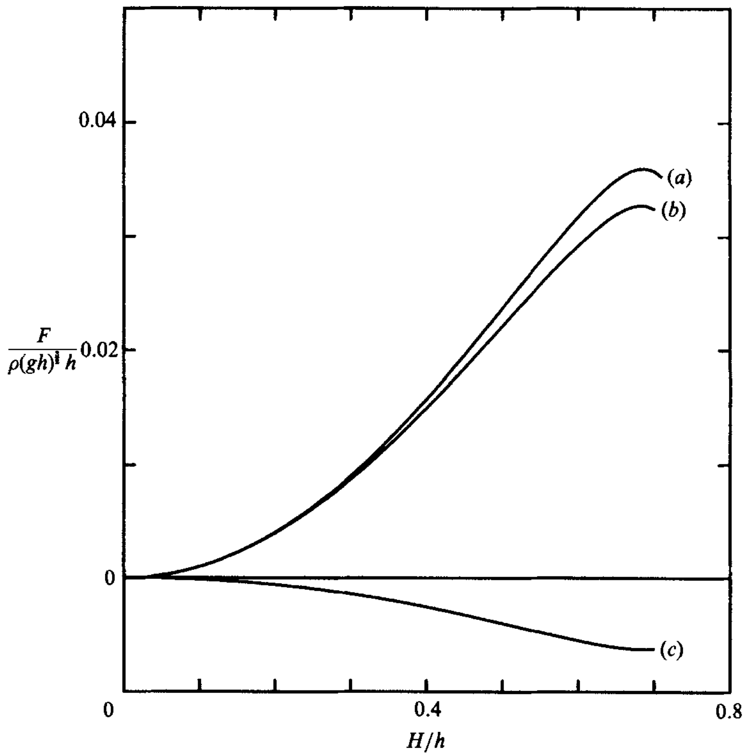

FIGURE 2. Dimensionless mean energy flux $F/(\rho(gh)^{\frac{1}{2}}h)$ as a function of dimensionless wave height H/h for a dimensionless period $(\lambda/c)(g/h)^{\frac{1}{2}} = 10$. (*a*) Stokes's first definition of phase velocity, equation (5.16) of Cokelet (1977) and equation (23); (*b*) Stokes's second definition of phase velocity, equation (23); (*c*) Stokes's second definition of phase velocity, equation (5.16) of Cokelet (1977).

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Note added in proof. After completion of this work, the author became aware that the same relationships for a general inertial frame of reference are given in Appendix IV of Sobey *et al.* (1987). They do not mention explicitly that their relationships correct those given by Cokelet (1977).

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